

Is Time a Continuum of Instants?

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The determinacy of physical quantities

We normally assume the determinacy of all physical quantities, that is, that every physical quantity must have, in reality, an absolutely determinate magnitude, represented, relatively to a given unit, by a real number. This determinacy principle embodies a *realist* view, in that it postulates states of affairs as subsisting independently of our knowledge of them. If physical theory represents quantities of a given type, such as energy, as quantized, and so capable of assuming as values only multiples of the quantum, we shall have a ground for regarding such quantities as having determinate magnitudes, representable, in terms of a suitable unit, by rational numbers in all cases. The determinacy principle applies, however, to quantities of every kind, including continuous ones such as temporal duration and spatial distance which we do not suppose to be quantized.

The determinacy principle recognizes the magnitude of a continuous quantity as representable, in terms of any given unit, by a real number, which may be rational or irrational. We *cannot* in principle ever know the precise magnitude of any continuous quantity. Hence the determinacy principle embodies, not merely a realist, but a *super-realist* metaphysics, in that it postulates states of affairs that subsist independently of even the theoretical possibility of our knowing of them. Why does such a conception appear compelling to us?

The magnitude of a quantity is the ratio of the quantity to a unit quantity of the same type (mass, length, etc.). As such, it is naturally specified by means of a rational number: we say that an event lasted for one and a half hours, that a place is two-thirds of a mile distant, that something weighs two and a quarter pounds. But when we aim at greater precision, we specify the magnitude with a margin of error: we determine it to within an interval with rational end-points. It is banal to say that we cannot in principle do better than this: we can never, by measurement, identify any specific real number as giving the magnitude of the quantity in terms of an assigned unit.

Why, then, do we not suppose that that is all the fact of the matter: that, in reality, nothing more holds good of the magnitude than that it lies within some such interval? The obstacle to our resting content with this is that there is not usually any known limit to the precision of our measurements. If we ever knew, of a certain technique of

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measurement, that it was in principle impossible that we should ever measure the magnitudes of quantities of that kind more accurately, we might be willing to hold that those quantities possessed in nature magnitudes of which no more precise allocation could be made than to say that they lay within intervals whose length corresponded to the margins of error of that method of measurement. We do not feel disposed to say this because we do not usually know that we have reached the limit of precision in measuring quantities of a given kind, and particularly not of temporal duration or of spatial distance: we constantly refine our techniques of measurement.

The assumption that the magnitude of each quantity is determined in reality, relatively to a given unit, by a specific real number involves the assumption that we could progressively refine our techniques of measurement indefinitely. The intervals within which we determined the magnitude by these ever more precise measurements would be nested: we assume that they would converge to a point in the real line.

The assumption that we could refine our techniques of measurement indefinitely, and that it is determinate what their results would be in any given case, is highly speculative. It embodies two realist presuppositions: first, that there is a determinate answer to hypothetical questions about what the results of ever more precise measurements would be, were they to be made; and, secondly, that an infinite process will yield a determinate outcome—in this case, the limit of an infinite monotonic sequence extending into the future. But even these realist principles, taken together, will not of themselves yield the determinacy principle—the assumption, namely, that the magnitude of each continuous quantity is determined in reality by a real number. Given that each member of the infinite sequence of measurements is correct, within the stated margin of error, the sequence of nested intervals that they yield will necessarily converge: but they might converge to an interval of the real line, rather than to a point on it. The assumption of absolutely determinate magnitudes cannot be derived from the imagined infinite sequence of ever more exact measurements. Rather, it rests on a metaphysical super-realism that conceives of physical reality as completely determinate and independent of our capacity to discover it.

The classical model

Applied to temporal duration, the determinacy principle grounds our model of time: we conceive of it on the analogy of the classical

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continuum of real numbers. This model is deeply engrained in our thinking: even in the thinking of those who could give no good account of the mathematical conception of the continuum. The continuum, classically conceived, is composed of its points, the real numbers, each of which exists independently of the others: each represents a determinate position on the rational line, a point on that line, if it is rational, a dimensionless gap in that line, if it is irrational. The ordering of real numbers by magnitude is a dense, complete linear ordering. According to the model which conceives of time by analogy with the classical continuum of real numbers, time is composed of durationless instants, arranged in a dense linear ordering; since they are durationless, no change or motion takes place within any instant. The ordering being dense, there is between any two instants another instant: there is therefore no such thing as the *next* instant after a given one. Moreover, time is *continuous*: there are therefore no gaps in the sequence of instants.

It requires a little mathematical sophistication to formulate this last principle. After all, it took many centuries before mathematicians attained a formulation of the intuitive conception of the continuity of the real line. The mathematically unsophisticated assume temporal instants to be ordered continuously just as the real numbers are: the fact that they cannot characterize this notion of continuity does not detract from the soundness of the claim that they, in common with the more sophisticated, conceive of time after the model of the real numbers. Continuity in this context means that the ordering of instants by temporal precedence is complete. That is, if a set of instants has an upper bound, there being an instant later than any in the set, then it has a least upper bound: there is an instant later than every instant in the set, or identical with one of them, and earlier than every other instant later than every instant in the set.

Duration, like any other quantity, must be measured relatively to a unit, say a second. We do not suppose that the duration of every temporal interval can be given in seconds by a rational number: to do so would be to allow for gaps in time, just as there is a gap in the rational line between numbers whose square is less than 2 and those whose square is greater than 2. But we do assume that every temporal interval has a precise duration, whose length, if exactly specified, would be given in terms of seconds by a real number, rational or irrational. Moreover, we think in the same way about *every* physical quantity. We conceive of every such quantity as having a precise magnitude, given, in terms of an appropriate unit, by a real number. We cannot determine this magnitude more closely than to within some approximation: but in reality, it is completely precise.

That is our engrained conception of time, of space and of physical reality generally; let us name it 'the classical model'. Space may be non-Euclidean, and may have more than three dimensions: but we still think that a precise position in space must be given by some n -tuple of real numbers, where n is the number of spatial dimensions. It is true that the molecular structure of matter and the quantization of energy, electric charge, etc., have dented the classical model as it applies to physical quantities other than space and time; but we tend to think of these as restrictions imposed by the laws of physics upon the magnitudes that those quantities can have, rather than limitations on our *concepts* of the quantities in question.

According to the classical model, the history of the physical universe is constituted by its states at all instants. Relativity teaches us that there is no unique canonical way to slice space-time into temporal cross-sections, but this does not refute the model: any particular frame of reference will determine a particular cross-sectioning into instantaneous states of the universe, and these will together constitute the history of the universe. In what does an instantaneous state of the universe consist? It consists in an assignment to all independent physical quantities of their magnitudes at the instant in question. Relying on the classical model, we conceive of such magnitudes, relatively to chosen units, on the analogy of real-valued functions on the real numbers: the argument t of such a function $f(t)$ gives the instant, measured in (say) seconds from some chosen temporal origin, and the value of the function is the magnitude of the quantity in terms of some unit.

Just as the real line is classically conceived as composed of its points, so a function from real numbers to real numbers is conceived as constituted by its value for each real number as argument. So long as we know no more than that it *is* a function defined for each real number, we have no reason to assume that there is any particular connection between its value for any given argument and its values for other arguments; we shall have such a reason only if we know that the function has some particular character, such as that it is everywhere continuous or everywhere differentiable. And this is how we think of physical reality: the state of the universe at any one instant is *logically* independent of its states at all other instants. It may not be independent in the light of the laws of physics: those laws may require that the previous states of the universe impose a restriction upon a subsequent instantaneous state. But this is a matter of the contingent laws which happen to govern the evolution of the universe: *logically*, each instantaneous state is independent of every other. This was precisely what Hume meant when he enunciated his celebrated dictum that 'all events are loose and separate'.

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Now from where does the classical model come? From reality or from our minds? We do not derive it from reality, or from our experience of reality. We impose it upon reality. The mathematical model comes first, even for those who grasp it only inchoately: it comes from our minds, and we apply it in thinking about reality. And the fit is very imperfect.

The first person, to my far from extensive knowledge, to have recognized this imperfect fit was St. Augustine. The thought that the history of the universe is constituted by its total state at every instant can be expressed thus: the present forms the substance of the world. This does not mean that there are no truths save those stating what holds good *now*. It means that the only other things which are true are those that state what held good or will hold good at some particular instant: those that state that of which it was or will be true to say, 'It holds good now'. The past is that which *has been* present, the future that which *will be* present. So there cannot be either a past or a future unless there is, independently of past or future, such a thing as how things are now.

Augustine, remarking that the past is no more and that the future has not yet come to be, asked how, then, the present could exist. For the present is a mere boundary between the past and the future. Since an instant has no duration, it is a mere boundary between the preceding time and the subsequent time; an instantaneous state is a mere boundary between the preceding and the subsequent course of events. But, Augustine observed, a boundary exists only in virtue of the existence of that which it bounds. There could not be a line, straight or curved, unless there were regions which it demarcated; there could not be a surface, plane or curved, unless there were some three-dimensional volume of which it was the surface.

It thus seems that the instantaneous state exists only in virtue of its being a momentary stage in some sequence of events that occupies a temporal interval of some duration. It must be the sequence of events that gives substance to the instantaneous states, not the instantaneous states that together give substance to the sequence.

Other instances of the imperfect fit between the classical model and physical reality arise from Hume's idea that instantaneous states are all logically independent of one another, or, otherwise expressed, from treating the magnitude of a physical quantity at an instant as given by a function from real numbers to real numbers. So understood, there is no reason provided by the *concept* of a physical quantity why its magnitude should change continuously: if it is constrained to change continuously, it can only be so constrained by the laws of physics, and not by conceptual necessity.

We need a little care in formulating Hume's doctrine that what

holds good at any one instant is logically independent of what holds good at any other instant. When 'what holds good' is taken to be the velocity of an object (or the rate of change of any other quantity), we can readily repudiate Hume's doctrine, since to assign a velocity to an object at a given instant is not simply to characterize the state of the universe at that instant. This is of course because velocity is defined as the derivative of distance with respect to time: hence to assign a velocity to an object at a certain instant is not to speak just about how things are at that instant, but to say something about how they are during intervals containing that instant, although not about any particular such interval.

It is not possible, however, for every quantity to be defined in terms of other quantities, nor, in particular, as the derivatives of such quantities with respect to time (or any other variable). There must be certain *basic quantities*, such as spatial position, mass and electric charge, not definable in terms of other quantities. The complete history of the physical universe over a closed temporal interval would in principle be completely characterized, on a more exact formulation of the classical model, by an assignment of the magnitudes of the basic quantities at each instant in that interval. Let us call such an assignment for any one instant an 'instantaneous physical state-description'. For such an assignment, no reason could be offered, by appeal to the classical model, why Hume should not have been right: there would be no *logical* connection between the physical state-description for any one instant and that for any other.

If a discontinuity in the magnitude of a basic quantity is not merely contrary to the laws of physics, but *conceptually* abhorrent, then Hume's doctrine is false, and the classical model of physical reality incorrect. But *is* such a discontinuity conceptually abhorrent? I do not think that it always is. Consider what is called a jump discontinuity. We can, in a fairly obvious way, define what is meant by saying that a function $f(x)$ approaches a limit q as x approaches u from the left, and that it approaches a limit r as x approaches u from the right. If f is continuous at $x = u$, then indeed $f(u) = q = r$. But suppose that $q \neq r$, and that either $f(u) = q$ or $f(u) = r$. Then f will have a jump discontinuity at $x = u$.

I do not believe that an abrupt discontinuity in the magnitude of a physical quantity is in itself conceptually abhorrent; on the contrary, common sense readily credits such abrupt discontinuities. Consider intensity of illumination. The illumination of a surface may gradually dim until it vanishes altogether. But, to gross observation, it may also abruptly vanish as when a candle, the only source of illumination, is extinguished. There seems no intrinsic absurdity in that: we speak, indeed, of something's 'going out like a light'.

What we unreflectively suppose is that the intensity of illumination goes instantaneously from some positive value to 0, where it remains for some further time.

That supposition would presumably be represented on the classical model by a function $f(t)$ giving the intensity of illumination at each time t within the appropriate interval, and having a jump discontinuity at $t = t_0$, for some t_0 within that interval. But here the classical model fails to fit intuition. According to that model there are *two* distinct such functions f and g with a jump discontinuity at $t = t_0$, with $f(t_0) = 0$ but $g(t_0)$ positive, while f and g agree for every other value of t within the interval. The model therefore represents the abrupt change as being one or other of two physically distinct events: one in which the illumination vanishes *at* the instant of change, the surface having a positive illumination at every instant *before* the change; and the other in which the surface continues to have a positive illumination at the instant of change, but 0 illumination for every instant in some interval *after* that instant.

Plainly, there are no two such distinct physical possibilities: nothing could determine whether the surface had zero or positive illumination at the precise instant of change, and we cannot conceive of there being any genuine distinction between the two cases. Here the classical model provides a means of differentiating between two physically different states of affairs which cannot possibly correspond to any distinction in physical reality.

Another type of simple discontinuity is a *removable* discontinuity. This is illustrated by a function f such that, for every x in the closed interval $[0, 2]$, $f(x) = x^2$, save that $f(1) = 2$. A discontinuity in the magnitude of a physical quantity that would be represented by a function exhibiting a discontinuity of this kind is quite evidently absurd: it cannot correspond to any physical reality. An example would be a pair of objects which, throughout a certain interval, were exactly 2 cm apart, save at one particular instant in that interval, when they were 4 cm apart. Our conception of physical quantities is plainly such that this supposition makes no sense. Yet the classical model allows it a sense: according to it, it is barred, if it is barred, only by the laws of physics, and not by conceptual necessity. The classical model supplies descriptions for states of affairs which, being conceptually impossible, should admit no description.

Yet another kind of discontinuity is exemplified by the frequently imagined example of a body which oscillates with increasing rapidity in a plane between a position R 1 cm to the right of a point M and a position L 1 cm to its left. It begins by swinging from M to L in $1/3$ minute, then from L to M in $1/6$ min, then from M to R in $1/10$ min, the n -th swing taking $2/(n+1)(n+2)$ min. The sum of the

first n terms of the series $1/3 + 1/6 + 1/10 + \dots$ is $n/(n+2)$, so the series converges to 1 ; hence by 1 min after the start the body will have made infinitely many swings. Wherever we suppose it then to be, there will be a discontinuity in its spatial position at that instant. For if $f(t)$ gives its position in cm to the right of M t min after the start, f does not approach any limit as t approaches 1 from the left.

Those who have proposed this as a genuine physical possibility have been in the grip of the classical model, which allows us to define a function f satisfying the conditions of the example: say $f(t) = \sin(\pi/(t-1))$ for $0 \leq t < 1$. Those who have contemplated it from the standpoint of common sense have been convinced that it does *not* represent a genuine physical possibility, and they are surely right. The common-sense thought, that makes this fantasy conceptually abhorrent, is that what happens to the body for $t < 1$ does not tell us at all where it will be at the instant $t = 1$. Its position at that instant is completely indeterminate: it might be anywhere, so far as its past history goes. We do not suppose that events are as loose and separate as this.

Is this a good argument? I believe that it is. There might be some physical explanation for where the body was to be found at $t = 1$: but whatever force caused it to assume that position at $t = 1$ can have operated only at that instant, and not before: its position at $t < 1$ can have had no influence on its position at $t = 1$. The oscillatory process had no natural termination: only a quite adventitious external force can explain why it ended as it did. Without any such external force, there could be no explanation at all for the final position of the body.

Suppose that $f(t) = 0$ for all t such that $1 \leq t \leq 2$: the body is at rest at position M during the closed interval $[1, 2]$. Then it might be said that, although the values of $f(t)$ for $0 \leq t < 1$ did not determine the value of $f(1)$, its values for t in the interval $[1, 2]$ did determine it: the objection is based on the mere prejudice that how things are is determined by what went before rather than by what is to come after. Well, then, let us reverse the example. Suppose that the position of the body during the interval $[0, 2]$ were given by the function $g(t)$, where $g(t) = f(2-t)$. In this new fantasy, the body is at rest at position M at any instant t with $0 \leq t < 1$: at any instant t such that $1 < t \leq 2$, then, however close to 1 t may be, the body will have made infinitely many swings, taking $1/3$ min to make the last one. This second fantasy evidently conflicts with our whole conception of what it is for a body to occupy a position in space at a given time; and yet it is obviously as easy to define a suitable function g as to define the original function f . Discontinuities of these kinds—in fact, all discontinuities save simple jump dis-

continuities—are to be rejected as impossible on pure conceptual grounds.

The classical model is to be rejected, because it fails to provide any explanation of why what appears to intuition to be impossible should be impossible. It allows as possibilities what reason rules out, and leaves it to the contingent laws of physics to rule out what a good model of physical reality would not even be able to describe.

Fuzzy realism

How can we arrive at a more satisfactory model of time and of other physical quantities? Such a model should render conceptually abhorrent discontinuous changes impossible to describe, and eliminate any distinction between diverse descriptions of abrupt changes.

Plainly, a better model will represent each physical quantity as fully describable by specifying an interval within which its magnitude lies. The interval will not merely represent the best approximation that we can achieve by measuring it: it will constitute the magnitude of the quantity *as it is in reality*. Let us therefore call it a *constitutive interval*. On this conception, physical reality is not fully determinate, in the sense that we are accustomed to take it to be. To adopt such a conception, it is not necessary to repudiate realism. It may be that the constitutive interval is narrower than any within which we shall ever locate the magnitude of the quantity by any technique of measurement we shall ever devise. So understood, the model is realist in that it does not explain the notion of a constitutive interval in terms of our methods of determining the magnitude of the quantity in question: it postulates that such an interval is what in reality that magnitude consists in, independently of whether we can discover it or not. I therefore label this model 'fuzzy realism'.

When we determine the instant at which an event began, we are measuring a temporal duration, namely from some origin until the beginning of the event. Hence, on the fuzzy realist model, the instant consists in a constitutive interval. Let us suppose that every constitutive temporal interval has the same length, given, in terms of seconds, say, by a rational number s ; for brevity, we may call such a temporal interval a 'moment'. We have no need to take the interval as containing irrational numbers, but may take it as being an open interval in the rational line, that is, as containing only rational numbers, but no smallest or greatest member, and, of course, as containing every rational lying between any two rationals that it

contains. There is, however, no reason to require that it have rational end-points. Since the model is a realist one, the objection that we could never by measuring it estimate a time as lying within an interval with irrational end-points is devoid of force.

Time, on this model, is not composed of durationless instants, densely ordered, but of overlapping intervals having some temporal extension. The duration of a moment, namely s seconds, will be extremely small—at most as small as twice the margin of error of our most accurate measurements of time. This length provides a natural unit of time, which we may call a ‘minim’ (mn): of course we may never be able to determine the length of a minim. The general form of a moment, measured in minims, will therefore be the set of rational numbers lying in an open interval $(m - 1/2, m + 1/2)$ of the real line, where m is the mid-point of the interval.

How, then, are we to represent the evolution of the magnitude of another physical quantity over time? The magnitude of any given physical quantity Q will likewise consist of a constitutive interval of some fixed length r , relative to an appropriate unit. This will yield a natural unit of Q , which I will call a ‘quintrum’ (qm). There will be a function F defined on moments T giving the magnitude of Q at T ; $F(T)$ will be an interval I of length 1 qm. The model is *not* one in which there are quanta of time, during which everything is static, like a cinema film. Change may be continuous, as on the classical model.

Where F is a function giving the magnitude of the quantity Q , in quitra, at a moment, we may require that F be continuous in the sense that, if S and T are overlapping moments, the intervals $F(S)$ and $F(T)$ will also overlap. This is not an arbitrary constraint, nor one imposed merely by the laws of physics, but a conceptual necessity, which may be recognized in the following way. If r belongs to a moment S , we may say that r is, at S , an *acceptable estimate* of the time; likewise, if q belongs to $F(S)$, q will be an acceptable estimate of the magnitude of Q . It therefore seems natural to require that, if r belongs to both S and T , there should be acceptable estimates of the magnitude of Q belonging to both constitutive intervals $F(S)$ and $F(T)$. It cannot be demanded that an acceptable estimate of the time will determine what is an acceptable estimate of the magnitude of Q ; but the demand that, if the same estimate of the time is acceptable both at the moment S and at the moment T , then there will be some estimate of the magnitude of Q that will be acceptable at both moments, needs to be met.

This requirement has the consequence that the quantity Q is subject to a maximum rate of change of 1 quintrum per minim. The argument is as follows. Let T be the moment

$$(m_T - 1/2, m_T + 1/2),$$

and let $F(T)$ be

$$(r_T - 1/2, r_T + 1/2).$$

Further, let T' be a moment preceding T , say

$$(m_{T'} - d - 1/2, m_{T'} - d + 1/2),$$

and let $F(T')$ be

$$(r_{T'} - 1/2, r_{T'} + 1/2)$$

Suppose that $r_{T'} > r_T$. T and T' will overlap when $d < 1$, so that $F(T)$ and $F(T')$ must then overlap. The condition for $F(T')$ to overlap $F(T)$ is that

$$r_{T'} - 1/2 < r_T + 1/2,$$

i.e. that $(r_{T'} - r_T) < 1$. The mean rate of change v of Q between T and T' is $(r_{T'} - r_T)/d$. As d approaches 1, so v will approach $(r_{T'} - r_T)$, whose maximum is 1. The argument is parallel if T precedes T' and if $r_T > r_{T'}$. Thus 1 qm/mn is a limit on the possible rate of change v . The limit can be attained. Let T_1 be $(m - 1/2, m + 1/2)$, T_0 be $(m - 3/2, m - 1/2)$ and T_2 be $(m + 1/2, m + 3/2)$. Suppose that $F(T_0) = (r - 3/2, r - 1/2)$, $F(T_2) = (r + 1/2, r + 3/2)$, and that $F(T)$ changes at a constant rate for T between T_0 and T_2 . Then for any such T , $F(T)$ will overlap $F(T_1)$. The rate of change of $F(T)$ between T_0 and T_2 will be 1 qm/mn.

Can there be change within a moment? On the classical model, a quantity Q cannot change *within* an instant t_0 , but Q may be changing *at* t_0 , in the sense that the value at $t = t_0$ of the derivative f' of the function $f(t)$ giving the magnitude of Q at each instant t is not 0. If the value of f' at $t=t_0$ is 0, we may say that Q is not changing at the instant t_0 ; but, of course, this may merely mean that Q attains its maximum or its minimum at that instant, and hence is changing within any neighbourhood of t_0 . We may treat the question similarly on the fuzzy realist model. Let R and S be distinct moments, with $R < S$; each will have a mid-point, say m_R and m_S . These will be real numbers; R will consist of the rationals within the open interval $(m_R - 1/2, m_R + 1/2)$ of the real line, and likewise for S . Q may change from the moment R to the moment S . The function $F(T)$ giving the magnitude of Q for any moment T between R and S is determined by the function $f(m)$ which gives the mid-point of $F(T)$ when m is the mid-point of T . The rate of change of Q is given by the derivatives f' of f : we may then say that Q is changing at T iff $f'(m)$ differs from 0. But we may look at the matter in a more fruitful way.

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Where $T = (m_T - 1/2, m_T + 1/2)$, Q may have different values at $(m_T - 1, m_T)$ and $(m_T, m_T + 1)$: in such a case we may say that Q is changing *within* the moment T .

Continuity under fuzzy realism

It is evident that, under fuzzy realism, there cannot be strictly discontinuous change. Discontinuities of the kind involving infinitely many oscillations of the same amplitude within a finite interval are ruled out, because the rate of change must tend to infinity towards the end of the interval, thus exceeding the maximum possible rate. Jump discontinuities and removable discontinuities may be simulated if the gap is sufficiently small. Suppose $r/2 < |p - q| < r$, where r is the length of a constitutive interval of the quantity Q. We want to simulate a jump discontinuity at an instant t from a steady value of p to a steady value of q . Where $T = (t - 1/2, t + 1/2)$, let us take $F(T)$ to be $((p + q)/2 - 1/2, (p + q)/2 + 1/2)$. Then $F(T)$ will contain both p and q ; the quantity Q will have both the value p and the value q within a single moment. In a similar way we can, under the same hypothesis about p and q , simulate a removable discontinuity according to which Q has the magnitude p at an instant t and the steady magnitude q throughout an interval following t and an interval preceding t . But these are only simulations: in truth, they are no more than very rapid changes. When $|p - q| > r$, neither the jump discontinuity nor the removable discontinuity can even be simulated: for the rate of change of Q would exceed the maximum possible. The fuzzy realist model does not accommodate the intuition that jump discontinuities are not conceptually abhorrent, but rules out all genuinely discontinuous change in the magnitude of a physical quantity.

A modification of fuzzy realism

On the version of fuzzy realism expounded above, every constitutive interval of a given quantity has the same length. But this may be unreasonable. A moment is given as the end-point of some physical process, a duration as the temporal extent of some process: may not the length of the interval that constitutes that moment, or of those that constitute the end-points of that duration, vary according to the nature of the process in question? We are familiar with the ambiguity of 'now' and 'at the present time' in such questions are 'What are you doing now (at the present time)?': the answer

depends on the kind of activity considered. We might similarly regard the length of a constitutive interval as depending on the kind of process of which it is taken as marking one or other end-point. The moment when a lecture began, the moment when a train started, the moment when a gun was fired, the moment when a neutron was emitted, measured in seconds from some origin, might, on this view, consist of intervals in the rational line of differing lengths.

According to the classical model, time is composed of durationless instants. According to the fuzzy realist model, in its unmodified form, it is composed of intervals all of the same length. This length must be very small, if it is to be smaller than the most accurate measurements that can be made: the period of oscillation of a cesium atom is calculated to be approximately $0.10878363 \times 10^{-9}$ seconds, or 0.10878363 nanoseconds (thousand millionths of a second). But if we allow that constitutive intervals may vary in length, some wholly including smaller ones, it does not seem that we can regard time as composed of such intervals. Rather, it seems that we must think of it as a continuum within which we can specify short stretches by reference to the beginning and end of physical processes. The length of a stretch so specified—a moment for the purpose in question—represents the most accurate determination it would make sense to make of the time of the chosen event. We may not have determined it so accurately; but there will be a limit to the intelligible accuracy we could attain.

What, then, does it mean to speak of time as being ‘composed’ of instants or of intervals? Time is the measure of change: its existence simply consists of there being functions giving the magnitudes of other quantities at different times. So time is given as the totality of possible arguments of such functions: instants on the classical model, constitutive intervals on the fuzzy realist one. The arguments of such functions are the basic temporal units: it is of them that time is composed. That is why, on the classical model, the values of any such function for distinct instants as arguments are logically independent, that is, Hume’s temporal atomism is logically true; likewise, on the fuzzy realist model, the values of such a function for two non-overlapping moments as arguments are logically independent. On this understanding of the metaphor of the units of which time is composed, however, time is composed of the constitutive intervals on the modified fuzzy realist model as well as on the unmodified one, even though in this case they are of varying length. Does this imply that the modification is incoherent? No. We can conceive of time according to the image of parallel strands, each composed of intervals of different lengths, according to the kind of physical process being clocked.

The constructive model

Suppose it thought that nothing can have a property it could not be observed to have, where observing it to have a property is understood in a liberal sense as any effective means, involving sensory perception, measurement, experiment and calculation, for establishing that it has that property. We can take this to mean that it has only those properties it has actually been or will at some time actually be observed in this sense to have; or that it has just those properties it *could* have been or could subsequently be observed at the relevant time to have, whether or not the observation was in fact or will in fact be made. For the present purpose it is of no account which of these, or any variation on them, is meant, or what criterion is used for saying that an observation 'could' have been made.

Such a view evidently calls for the use of the intuitionistic continuum of real numbers in place of the classical continuum. In intuitionistic mathematics, a real number is given by a sequence $\langle r_n \rangle$ of rationals that satisfies the Cauchy condition for convergence, understood constructively: that is, that for every negative power 2^{-k} , we can effectively find a term r_n of the sequence such that every subsequent term r_m differs from it by less than 2^{-k} . Such a Cauchy sequence of rationals is said to *generate* the real number to which it converges.

The intuitionistic theory of real numbers distinguishes itself both from its classical counterpart and from rival versions of constructive mathematics by its conception of infinite sequences. An infinite sequence of objects of whatever kind can always be thought of as obtained from some underlying sequence of natural numbers by means of a correlation law, which associates an object of the desired kind to each initial segment of the underlying sequence. An infinite sequence of natural numbers may be given by means of some effective rule determining the value of each term. But it may also be given by some process under which the value of each term is freely chosen, or chosen under some effective restriction. In examples given to illustrate the notion, the choice of each term is frequently determined by some unpredictable empirical event. Sequences so generated are called *free choice sequences*. If we want to consider all sequences subject to the same initial restriction, we can represent this restriction by means of a *spread law*, which determines effectively whether any given finite sequence of natural numbers is or is not admissible under this restriction: if it is inadmissible, it will not be an initial segment of any infinite sequence satisfying the restriction. The totality of infinite sequences every initial segment of which is admissible forms the *spread* determined

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by the given spread law. Thus diagram 1 illustrates the full ternary spread, which comprises all and only those sequences whose terms are all less than 3.

A correlation law will associate with an infinite sequence of natural numbers an infinite sequence of objects of some other kind, for instance rational numbers. The totality of the infinite sequences so correlated to the elements of a spread is called a *dressed spread*. Diagram 2 illustrates a dressed spread of sequences of rational numbers—those namely in the open interval $(0, 1)$ with powers of 2 as denominators—obtained from the full ternary spread by a suitable correlation law. The correlation law has been chosen so that the same rational number is correlated with different finite sequences in the ternary spread (initial segments of different infinite sequences). Thus $5/8$ is correlated both with $\langle 2, 0 \rangle$ and with $\langle 1, 2 \rangle$. Graphically, one may say that different paths in the tree may arrive at the same node. It will be evident that the elements of the dressed spread illustrated in Diagram 2 all satisfy the Cauchy condition and that, to every real number in the closed interval $[0, 1]$, some element of the spread, representing its binary expansion, converges.

A sequence given by means of an effective rule is considered an element of a given spread if all its initial segments are admissible;

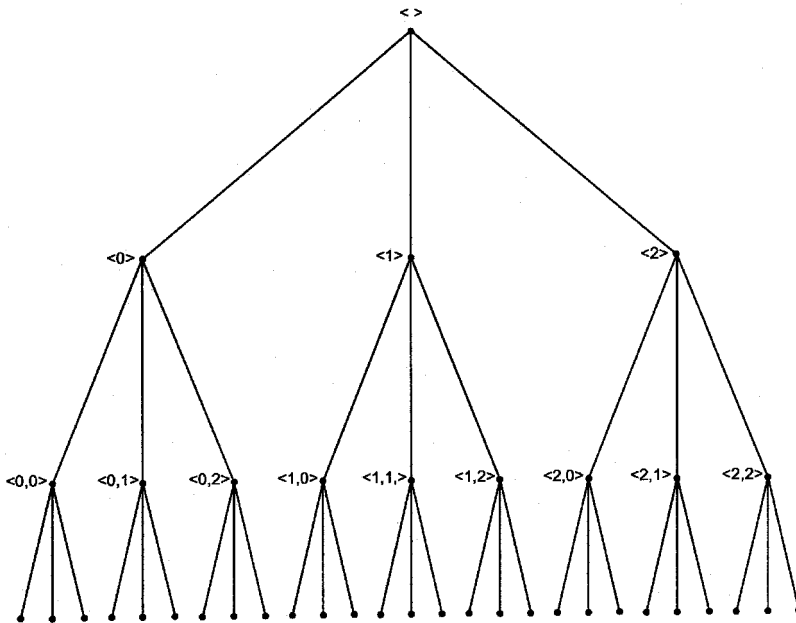


Diagram 1

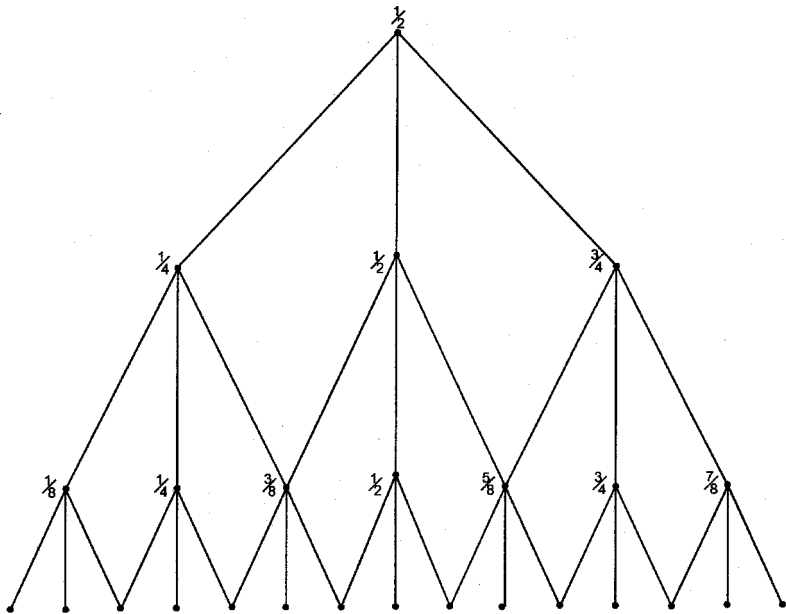


Diagram 2

but, for particular purposes, we may restrict our attention to *lawless sequences*, which are generated by free choices of terms unconstrained by any restriction imposed in advance or at any later stage, or to *lawless elements* of a given spread, those generated by choices constrained only by the spread law. For any practical purposes, it is better to take our variables as ranging over the lawless elements together with elements obtained from them by a continuous operation; here β will be said to have been obtained by a continuous operation from α if every term of β resulted by an effective method from some initial segment of α .

What is the point of considering sequences generated by free choices, or by empirical observations of unpredictable phenomena? These are *infinite* sequences, and from the intuitionistic standpoint they are not to be considered as capable of being completed and regarded as if every one of their terms was determinate. Rather, we can say about such a sequence only what could be known about it at some stage in the process of generating it. What we know about it is (i) how it is generated, for instance as subject to a certain spread law and a certain correlation law, as the result of a continuous operation on some other sequence, and by means of free choices made or given in some particular way, and (ii) some finite number of its

terms, i.e. the members of some initial segment of it. On the constructive model of physical quantities, the magnitude of any such quantity, relative to a given unit, is constituted by the real number determined by a choice sequence whose terms are given by successive measurements of it actually carried out or capable of having been carried out. We may regard each element of the dressed spread illustrated in Diagram 2—each infinite sequence of rational numbers given by a path in the tree—as representing the sequence of measurements, in some unit, of the magnitude of some particular quantity, individual or generic, that we shall actually make or that we shall be able at successive times to make; the unit has been chosen so that the magnitude of the quantity is known to be no greater than 1 unit. For this purpose, the inclusion in the domain of our sequence-variables of sequences derived by some continuous function from other sequences allows us to include as measurements of a quantity approximate values calculated in accordance with some law from the measurements made of some other quantity. We are assuming that each measurement is correct, within its margin of error, and that each is a refinement on the preceding one, to the extent given by the spread and correlation laws. The magnitude of the quantity, in the given unit, is then the real number to which the sequence of rationals determined by the measurements we make or could make converges. But we can never make any assertion whose truth depends upon *all* the terms of that sequence, since we can never know all those terms; any assertion we make must depend only upon an initial segment of the sequence, that is, upon the measurements made to date. We can express this by saying that the true value of the magnitude of the quantity depends upon the indefinite future, but that all that holds good of it at any given time is what has been determined to hold good of it by that time.

How, then, is the margin of error of a measurement represented in Diagram 2? There is an ambiguity in the notion of the margin of error. Construed as what we may call the *long-run* margin of error, it shows the limits within which the true value of the magnitude may lie; this is perhaps the standard interpretation. So understood, it does *not* give the limits between which the next measurement, correct but more accurate, may lie. These limits are given by what we may call the *next-step* margin of error, which must be smaller than the long-run margin. This point is independent of whether a constructive or classical view is adopted; consider the matter, for the moment, from a classical standpoint. A measurement is made, yielding a value of r units $\pm \delta$ units. Interpret the margin of error δ as a next-step margin. It is then possible that the next measurement will give the value $r - \delta$ units, with a new smaller margin of error $\pm \epsilon$.

The next measurement after that may then yield a value of $r - \delta - \epsilon \pm \zeta$: the true value of the magnitude is definitely smaller than $r - \delta$ units. Suppose, in Diagram 2, we have twice measured the magnitude of some quantity, obtaining the value $1/2$ each time: we are at the node corresponding to $\langle 1, 1 \rangle$ in the naked spread. The next-step margin of error is plainly $\pm 1/16$: the next measurement may give any of the values $7/16, 1/2, 9/16$. Suppose that in fact the next measurement gives the value $7/16$, and that thereafter the sequence continues to follow the leftmost path: this represents the sequence $\langle 1, 1, 0, 0, 0, \dots \rangle$ in the naked spread, which corresponds by the correlation law to the sequence $\langle 1/2, 1/2, 7/16, 13/32, 25/64, \dots \rangle$ in the dressed spread. This latter sequence converges to the number $3/8$, which may therefore be the ultimate value of the magnitude in question. Similarly, the next measurement might be $9/16$, and successive measurements might take us down the rightmost path, converging to the value $5/8$. The long-run margin of error of the second measurement of $1/2$ units was therefore $\pm 1/8$.

As before, the magnitude of any physical quantity other than temporal duration may be considered as given by a function defined on the time, measured as that elapsed since an event taken as determining a convenient temporal origin; the argument of the function is the real number to which the sequence of approximate measurements of the time elapsed converges. Problems of continuity do not arise on this model: every such function must be continuous. This follows from the famous, or notorious, theorem of Brouwer, that any function defined on every real number must be continuous (and on every real number in a closed interval uniformly continuous).

The constructive model does not represent time as composed of durationless instants corresponding to determinate real numbers, as the classical model does, nor of small constitutive intervals, as the unmodified fuzzy realist model does. Rather, somewhat like the modified version of realism, it represents time as a continuum which we can dissect into intervals whose end-points are the initiation and termination of physical processes. We can determine the end-points of such intervals as themselves much smaller intervals, these being our approximations to the instants at which they occurred. Such instants are indeed representable by real numbers, and it is on them that are defined the functions giving the magnitudes of other quantities at different times; in this respect all is as in the classical model, and we may, in the sense already stated, say that time is composed of such instants. But the constructive model differs from the classical one in that these instants are not precisely located: they are ideal constituents, not actual ones. We seek to locate the instants by ever more accurate measurements, and there

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is no limit in principle to this process. But determining the precise location of an instant is an infinite process, which can never be completed. Time is only notionally composed of instants, not actually so: instants are unattainable theoretical limits to the process of dissection. A function giving the magnitude of a physical quantity at a time, when applied to an approximation to the time—an interval enclosing the ultimate unattainable value—will, being continuous, yield an approximation to the magnitude of the quantity—an interval enclosing the ultimate unattainable value; applied to a closer approximation to the time, it will yield a closer approximation to the magnitude. A realist will say that this is a good description of *our* imperfect methods of determining instants and magnitudes, but that we must believe that the limits we cannot attain exist in reality, though known only to God. The constructivist asks *why* we should believe this: he does not think that reality contains, or that God creates, anything of which His rational creatures cannot in principle become aware.

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